Metric and Curvature in Gravitational Phase Space

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Abstract

At a fixed point in spacetime (say, x_0), gravitational phase space consists of the space of symmetric matrices $\{F^{ab}\}$ [corresponding to the canonical momentum $\pi^{ab}(x_0)$] and of symmetric matrices $\{G_{ab}\}$ [corresponding to the canonical metric $g_{ab}(x_0)$], where $1 \le a, b \le n$, and, crucially, the matrix $\{G_{ab}\}$ is necessarily positive definite, i.e. $\sum u^a G_{ab} u^b > 0$ whenever $\sum (u^a)^2 > 0$. In an alternative quantization procedure known as *Metrical Quantization*, the first and most important ingredient is the specification of a suitable metric on classical phase space. Our choice of phase space metrics, guided by a recent study of Affine Quantum Gravity, leads to gravitational phase space geometries which possess *constant scalar curvature* and may be regarded as higher dimensional analogs of the Poincaré plane, which applies when $n = 1$. This result is important because phase spaces endowed with such symmetry lead naturally via the procedures of Metrical Quantization to acceptable Hilbert spaces of high dimension.

I. Introduction

The canonical phase-space variables of classical gravity have been identified as the spatial metric with components $g_{ab}(x)\,[= g_{ba}(x)]$ and its canonical momentum with components $\pi^{ab}(x)\,[= \pi^{ba}(x)]$, where $a, b = 1, 2, \ldots, n$, in

an $(n+1)$-dimensional spacetime [1]. While the momentum variables are unrestricted in the values they may assume, the metric variables are restricted in domain by their physical interpretation so that the $(n \times n)$-matrix made from the metric components is positive definite for all x, i.e., $\{g_{ab}(x)\} > 0$. Stated otherwise, if u^a, $a = 1, \ldots, n$, denotes an arbitrary vector which satisfies $\Sigma\,(u^a)^2 > 0$, then we insist that $u^a\, g_{ab}(x)\, u^b > 0$ for all x. Therefore, while the momentum variables $\pi^{ab}(x)$ are elements of a linear vector space, the metric variables $g_{ab}(x)$ decidely do *not* form a linear vector space. These important physical facts will have a basic significance for our discussion.

Our use of the word "gravity" is strictly motivational and carries no implication that the metric and momentum variables satisfy either the equations of motion or the constraints that characterize Einstein's gravitational theory. Instead, our discussion applies to a more primitive stage, namely, preparing the kinematics of classical phase space to accomodate phase-space variables that have the domain limitations that characterize the usual gravitational variables. Our goal is to discuss metrics and their curvature on such phase-space manifolds that fully respect these important domain restrictions.

Indeed, our motivating interest in phase-space metrics arises from a special version of quantization. Traditionally, of course, a Riemannian structure on classical phase space is not normally invoked in quantization, but there is a little-known approach to quantization, called *Metrical Quantization* in which *the choice of a Riemannian metric on classical phase space is the first and key ingredient* in that procedure, and it is that approach which we have in mind [2].

To help the reader get a feel for metrical quantization, let us turn to a brief survey of this approach for a single degree of freedom system.

Metrical Quantization

As an illustration of this procedure, we discuss two simple, one degree of freedom examples. The traditional phase-space path integral, stripped of any specific dynamics, formally reads (in units where $\hbar = 1$)

$$\mathcal{M} \int e^{i\int p\,\dot{q}\,dt}\,\mathcal{D}p\,\mathcal{D}q\;, \tag{1}$$

where $\mathcal{M}$ represents a suitable normalization. This "integral" is notoriously ill defined. One way to regularize and thereby give meaning to this formal

expression is as follows:

$$\langle p'', q''|p', q'\rangle = \lim_{\nu\to\infty} \mathcal{N}_\nu \int e^{i\int p\,\dot q\,dt-(1/2\nu)\int(\dot p^2+\dot q^2)\,dt}\,\mathcal{D}p\,\mathcal{D}q\;, \tag{2}$$

where now $\mathcal{N}_\nu$ is a ν-dependent normalization factor. In fact, this latter expression can be given a rigorous and unambiguous mathematical meaning as

$$\langle p'', q''|p', q'\rangle = \lim_{\nu\to\infty} 2\pi\, e^{\nu T/2} \int e^{i\int p\,dq}\, d\mu_W^\nu(p,q) \tag{3}$$

in terms of a two-dimensional Wiener measure μ_W^ν on a flat phase space (note, $\dot p^2 + \dot q^2 \propto dp^2 + dq^2$), where the parameter ν denotes the diffusion constant. The result of this path integral over phase-space paths $p(t)$, $q(t)$, $0 < t < T$, is a kernel, which we have already called $\langle p'', q''|p', q'\rangle$, and which depends on the initial and final pinned values for each of the two paths, i.e., $p(0) = p'$, $q(0) = q'$ and $p(T) = p''$, $q(T) = q''$. As the notation is intended to suggest, the result of the path integral is in fact just the overlap of two *canonical coherent states* of the form

$$|p, q\rangle \equiv e^{-iq\,P}\, e^{ip\,Q}\, |0\rangle\;, \tag{4}$$

where the fiducial vector $|0\rangle$ is a unit vector that satisfies $(Q+iP)\,|0\rangle = 0$. As usual, Q and P denote self-adjoint Heisenberg operators that satisfy $[Q, P] = i\mathbb{1}$. It is important to understand that all the properties implicit in (4) are a *direct consequence* of the assumed form of the regularization—specifically, the form of the *phase-space metric*—introduced into (2). This fact follows because the metric determines the kernel, and through the GNS (Gel'fand, Naimark, and Segal) Theorem [3], the kernel determines the form of (4) up to unitary equivalence. Thus, the very choice of the metric, specific to a flat space expressed in Cartesian coordinates, has, through (2) and (3), led to an explicit quantization in terms of canonical Heisenberg variables P and Q as the basic kinematical operators.

It is noteworthy that a change of the metric from $d\sigma^2 = dp^2 + dq^2$ to $d\sigma^2 = \lambda^{-1} q^2\, dp^2 + \lambda q^{-2} dq^2$, $q > 0$, $\lambda > \frac{1}{2}$, leads to an *entirely different quantum kinematical structure*. We first observe that the geometry of the phase-space manifold under the second metric is a simply-connected space of constant scalar curvature $-2\lambda^{-1}$ (the *Poincaré plane*), and which, therefore, constitutes a distinctly different geometry than that of the flat case.

If we use the new metric to support the Brownian motion paths, then the corresponding phase-space path integral reads

$$\begin{aligned}\langle p'', q'' | p', q' \rangle &= \lim_{\nu \to \infty} \tilde{\mathcal{N}}_\nu \int e^{i \int p\, \dot{q}\, dt - (1/2\nu) \int (\lambda^{-1} q^2 \dot{p}^2 + \lambda\, q^{-2} \dot{q}^2)\, dt}\, \mathcal{D}p\, \mathcal{D}q \\ &= \lim_{\nu \to \infty} 2\pi [1 - (2\lambda)^{-1}]^{-1} e^{\nu T/2} \int e^{i \int p\, dq}\, d\tilde{\mu}^\nu_W(p, q)\ . \qquad (5)\end{aligned}$$

Here $\tilde{\mu}^\nu_W$ is also a Wiener measure on a two-dimensional manifold but now a manifold of constant scalar curvature. Observe that the second form of (5) has a rigorous and unambiguous meaning. The remarkable fact, thanks again to the GNS Theorem, is that the phase-space path integral (5) determines a kernel that is given this time by the overlap of *affine coherent states* as defined by the expression

$$|p, q\rangle \equiv e^{ip(Q-q)}\, e^{-i \ln(q)\, D}\, |0\rangle\ , \qquad q > 0\ , \qquad (6)$$

where $[Q, D] = iQ$, $Q > 0$, and $|0\rangle$ is a normalized solution of $[D - i\lambda(Q - 1)]\,|0\rangle = 0$. Thus the second metric choice has resulted in an explicit quantization in terms of affine Lie algebra variables D and Q as the basic kinematical operators (rather than the more common P and Q).

Finally, let us observe that it is possible to consider the expression

$$\lim_{\nu \to \infty} \mathcal{N}'_\nu \int e^{i \int p\, \dot{q}\, dt - (1/2\nu) \int [d\sigma(p,q)^2/dt^2]\, dt}\, \mathcal{D}p\, \mathcal{D}q\ , \qquad (7)$$

for a general Riemannian metric

$$d\sigma(p, q)^2 \equiv A(p, q)\, dp^2 + 2\, B(p, q)\, dp\, dq + C(p, q)\, dq^2\ , \qquad (8)$$

where $A > 0$, $C > 0$, and $AC > B^2$, and to ask what quantum Hilbert space might then apply. Although we have couched the question in terms of a phase-space path integral, by the Feynman-Kac Theorem we can reformulate the question in terms of a second order partial differential equation. For example, the path integral (7) may be interpreted in terms of a two-dimensional particle whose kinetic energy is $(1/2\nu) \int [d\sigma(p, q)^2/dt^2]\, dt$ moving in the presence of a uniform magnetic field. The dimensionality of the quantum Hilbert space is the same dimensionality as the degeneracy of the lowest level (ground state) of the associated Hamiltonian.

Generalized Affine Coherent States and Gravitational Phase Space

Motivated by the desire for a quantum kinematical framework in which the aforementioned algebraic restrictions on the spacial portion of the metric of general relativity are automatically respected [4], we have recently examined an interesting matrix generalization of the affine algebra in which the spectrum of any line element is *manifestly positive*. This algebra generates a family of coherent states (*generalized affine coherent states*) whose overlap function may be represented as [cf. (5)]

$$\begin{aligned}&\langle F'', G''|F', G'\rangle \\ &\quad = \lim_{\nu\to\infty} \mathcal{M}_\nu \int e^{i\int\{-\mathrm{tr}(G\dot{F})\}dt \,-\, (1/2\nu)\int\{\lambda^{-1}\mathrm{tr}[(G\dot{F})^2] \,+\, \lambda\,\mathrm{tr}[(G^{-1}\dot{G})^2]\}\,dt} \\ &\qquad\qquad\qquad\qquad \times \prod_{j\le k} \mathcal{D}F^{jk}\,\mathcal{D}G_{jk}, \end{aligned} \tag{9}$$

where now $F \equiv \{F^{jk}\}$ is an $n\times n$ symmetric matrix, $G \equiv \{G_{jk}\}$ is an $n\times n$, *positive*, symmetric matrix, and λ is a scaling parameter; for the precise meaning of (9), as well as a discussion of related matters, see [5]. The phase space metric appearing in the regularization factor in (9) is described by

$$\begin{aligned} d\sigma^2 &= \lambda^{-1}\mathrm{tr}[(G\,dF)^2] + \lambda\,\mathrm{tr}[(G^{-1}\,dG)^2] \\ &= \lambda^{-1}G_{kl}\,G_{mj}dF^{jk}\,dF^{lm} + \lambda\,G^{kl}\,G^{mj}dG_{jk}\,dG_{lm}, \end{aligned} \tag{10}$$

where the indices j, k, l, m, take on values from 1 to n.[1] The nonzero components of the phase space metric may thus be written explicity as

$$\begin{aligned} g_{F^{jk}F^{lm}} &= \tfrac{1}{2}\lambda^{-1}\,(G_{kl}\,G_{mj} + G_{jl}\,G_{mk}), \\ g_{G_{jk}G_{lm}} &= \tfrac{1}{2}\lambda\,(G^{kl}G^{mj} + G^{jl}\,G^{mk}), \end{aligned} \tag{11}$$

[1]It is also of interest to note that the metric in (10) is formally identical to

$$||d|F,G\rangle||^2 - |\langle F,G|d|F,G\rangle|^2,$$

the infinitesimal ray metric on the corresponding Hilbert Space. This relation makes clear that $d\sigma^2$ is invariant under all right translations induced by the affine group that defines the coherent states in [5], and therefore characterizes a homogeneous space.

and its determinant is a function only of the dimensionality, namely,

$$\det g = 2^{n(n-1)}. \tag{12}$$

We shall refer to the $n(n+1)$-dimensional manifold spanned by the coordinates $F^{jk}(j \leq k)$ and $G_{lm}(l \leq m)$, with $\{G_{lm}\} > 0$, as *gravitational phase space*. In what follows, we shall also implicitly include a metric on gravitational phase space as part of its definition.

From the discussion in the previous section it should be clear that the geometry of gravitational phase space is of crucial importance in determining the nature of the associated Hilbert space as determined by (9). We now turn to an investigation of this geometry.

II. Curvature Calculations

In this section we outline the calculation leading to the scalar curvature of a gravitational phase space of general dimensionality whose metric is described by (10). It first proves convenient, however, not to enforce the customary symmetry conditions $F^{kj} = F^{jk}$ and $G_{ml} = G_{lm}$, thus constructing a $2n^2$-dimensional manifold which we shall refer to as the *extended phase space*. We show that the scalar curvature[2]

$$R_E = -2n^3\lambda^{-1}. \tag{13}$$

The corresponding result for the space of symmetric matrices (where we again enforce $F^{kj} = F^{jk}$ and $G_{ml} = G_{lm}$), may be extracted from the non-symmetric calculation via a careful process of symmetrization, whereby all the terms associated with skew-symmetric directions on the manifold are eliminated. We thus show that the scalar curvature of gravitational phase space is

$$R = -\tfrac{1}{2}n(n+1)^2\lambda^{-1}. \tag{14}$$

The correctness of the results in (13) and (14) may be verified computationally for at least the first few values of n.

[2]For convenience, we follow a sign convention for the scalar curvature in which the scalar curvature of the Poincaré plane is *negative*. See [6] for a full discussion.

Curvature of the Extended Phase Space

We begin by discussing the geometry of the $2n^2$-dimensional manifold whose coordinates may be taken to be the n^2 elements of the matrix F together with the n^2 elements of the matrix G. At this point, we treat F^{jk} and F^{kj} (and likewise G_{lm} and G_{ml}) as completely independent coordinates - the restriction to the submanifold of symmetric matrices is descibed in the following subsection.

The connection coefficients associated with the metric in (10) may be calculated by using the standard trick of considering the dynamics of a classical free particle moving in the corresponding $2n^2$-dimensional geometry. The motion of such a particle is governed by a Lagrangian of the form

$$L = \tfrac{1}{2}\mathrm{tr}[\lambda^{-1}(G\dot{F})^2 + \lambda(G^{-1}\dot{G})^2], \tag{15}$$

leading to equations of motion for F and G given by

$$\begin{aligned} \ddot{F} &= -(\dot{F}\dot{G}G^{-1} + G^{-1}\dot{G}\dot{F}), \\ \ddot{G} &= \lambda^{-2}G\dot{F}G\dot{F}G + \dot{G}G^{-1}\dot{G}. \end{aligned} \tag{16}$$

A re-interpretation of (16) as geodesic equations quickly reveals the form of the nonzero symmetric connection coefficients as

$$\Gamma^{G_{jk}}_{F^{ab}F^{cd}} = \Gamma^{G_{jk}}_{F^{cd}F^{ab}} = -\tfrac{1}{2}\lambda^{-2}(G_{ja}G_{bc}G_{dk} + G_{jc}G_{da}G_{bk}), \tag{17}$$

$$\Gamma^{F^{jk}}_{G_{ab}F^{cd}} = \Gamma^{F^{jk}}_{F^{cd}G_{ab}} = \tfrac{1}{2}(\delta^k_d\delta^b_c G^{ja} + \delta^j_c\delta^a_d G^{bk}), \tag{18}$$

$$\Gamma^{G_{jk}}_{G_{ab}G_{cd}} = \Gamma^{G_{jk}}_{G_{cd}G_{ab}} = -\tfrac{1}{2}(G^{bc}\delta^a_j\delta^d_k + G^{da}\delta^c_j\delta^b_k). \tag{19}$$

Ricci tensor elements may be constructed from the connection coefficients via the well-known relation [7]

$$R_{\alpha\beta} = \partial_\gamma\Gamma^\gamma_{\alpha\beta} - \partial_\beta\Gamma^\gamma_{\gamma\alpha} + \Gamma^\gamma_{\gamma\delta}\Gamma^\delta_{\alpha\beta} - \Gamma^\delta_{\beta\gamma}\Gamma^\gamma_{\delta\alpha}. \tag{20}$$

The second and third terms on the right hand side of (20) vanish as a consequence of the constant nature of the determinant of the metric in (10). Evaluation of the remaining terms yields

$$R_{F^{jk}F^{lm}} = \frac{\partial}{\partial G_{ab}}\Gamma^{G_{ab}}_{F^{jk}F^{lm}} - \Gamma^{G_{ab}}_{F^{cd}F^{jk}}\Gamma^{F^{cd}}_{F^{lm}G_{ab}} - \Gamma^{F^{ab}}_{G_{cd}F^{jk}}\Gamma^{G_{cd}}_{F^{lm}F^{ab}}$$

$$
\begin{aligned}
&= -n\lambda^{-2}G_{mj}G_{kl}, &(21)\\
R_{G_{jk}G_{lm}} &= \frac{\partial}{\partial G_{ab}}\Gamma^{G_{ab}}_{G_{jk}G_{lm}} - \Gamma^{G_{ab}}_{G_{cd}G_{jk}}\Gamma^{G_{cd}}_{G_{lm}G_{ab}} - \Gamma^{F^{ab}}_{F^{cd}G_{jk}}\Gamma^{F^{cd}}_{G_{lm}F^{ab}}\\
&= -nG^{mj}G^{kl}. &(22)
\end{aligned}
$$

Finally, contraction with the inverse metric then yields the scalar curvature of the extended phase space manifold,

$$
\begin{aligned}
R_E &= g^{F^{jk}F^{lm}}R_{F^{jk}F^{lm}} + g^{G_{jk}G_{lm}}R_{G_{jk}G_{lm}}\\
&= \lambda G^{jm}G^{lk}R_{F^{jk}F^{lm}} + \lambda^{-1}G_{jm}G_{lk}R_{G_{jk}G_{lm}}\\
&= (-n^3\lambda^{-1}) + (-n^3\lambda^{-1})\\
&= -2n^3\lambda^{-1}. &(23)
\end{aligned}
$$

It is worth pointing out that the F-G crossterms of (18) play a vital role in the calculation above - the scalar curvature of the corresponding n^2-dimensional manifold involving only the elements of the matrix G (and not those of F) turns out to be $-\frac{1}{2}n(n^2-1)\lambda^{-1}$, not simply $-n^3\lambda^{-1}$, as a cursory glance at (23) might suggest.

Curvature of the Phase Space of Symmetric Matrices

We now restrict attention to the submanifold of symmetric matrices defined by the restrictions $F^{kj} \equiv F^{jk}$ and $G_{ml} \equiv G_{lm}$. The metric of (10) again applies, subject of course to the stipulations that the variations dF and dG should likewise satisfy $dF^{kj} \equiv dF^{jk}$ and $dG_{ml} \equiv dG_{lm}$.

The curvature of the submanifold of symmetric matrices may again be obtained starting from (20). This time however it is necessary to employ *symmetric contractions* in (21) and (22) in order to filter out those contributions present in the previous calculation involving skew-symmetric directions. This process yields the Ricci tensor elements

$$
\begin{aligned}
R_{F^{(jk)}F^{(lm)}} &= -\tfrac{1}{2}(n+1)\lambda^{-2}G_{(m(j}G_{k)l)} &(24)\\
R_{G_{(jk)}G_{(lm)}} &= -\tfrac{1}{2}(n+1)G^{(m(j}G^{k)l)} &(25)
\end{aligned}
$$

and scalar curvature

$$
\begin{aligned}
R &= g^{F^{(jk)}F^{(lm)}}R_{F^{(jk)}F^{(lm)}} + g^{G_{(jk)}G_{(lm)}}R_{G_{(jk)}G_{(lm)}}\\
&= [-n(n+1)^2\lambda^{-1}/4] + [-n(n+1)^2\lambda^{-1}/4]\\
&= -\tfrac{1}{2}n(n+1)^2\lambda^{-1}. &(26)
\end{aligned}
$$

The two-dimensional result, $R(n=1) = R_E(n=1) = -2\lambda^{-1}$, will be recognized as the constant negative scalar curvature of the Poincaré plane. Notice that the scalar curvature of the generalized higher-dimensional phase space manifolds we have described remains constant for all n.

We emphasize that F-G crossterms of (18) are again important in the derivation of (26); in particular, the scalar curvature of the corresponding submanifold involving only the symmetric matrix G is $-n(n-1)(n+2)/8\lambda$.

III. Conclusion

The phase space metrics described by (10) define gravitational phase spaces of constant scalar curvature, the key ingredient necessary to induce, via (9), the infinite dimensional Hilbert spaces associated with the generalized affine algebra discussed in [5].

The physical concept behind our continuous-time regularization procedure involves the motion of a free particle on a curved space (in which our phase space is regarded as the configuration space). Quantization of such a system leads to the Laplace-Beltrami operator plus a possible additional term of order $\hbar^2$ proportional to the *scalar curvature* (see, e.g., [8]). As our scalar curvature is constant, this term represents a harmless factor in the integral (9) that can be included in the overall normalization. Higher symmetry, such as that represented by constant *sectional* curvature, is not required to obtain this result. In fact, although our metrics (10) do indeed possess the high degree of symmetry inherited from their group definition, they do not (for $n \geq 2$) define spaces of constant sectional curvature, since the identification of the various coordinates as elements of a positive definite matrix implies a lack of isotropy on the manifold.

The type of argument we have presented can also be used to disqualify a large class of candidate phase space metrics from serious consideration. For example, a few minutes with a tensor manipulation program will be enough to convince the reader that the plausible-looking phase space metric [9] described by

$$d\sigma^2 = (\det G)^{-1/2}\,\mathrm{tr}[(G\,dF)^2] + (\det G)^{1/2}\,\mathrm{tr}[(G^{-1}\,dG)^2] \tag{27}$$

does *not* lead to a gravitational phase space of constant scalar curvature, and therefore cannot generate via an analog of (9) anything other than a Hilbert

space of trivial dimensionality.

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